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UPPER AND LOWER BOUNDS OF BUCKLING LOADS

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Abstract—Upper and lower bounds of buckling load for a nonuniform elastic column under conservative loading are considered. Compatible admissible moment and displacement functions are expressed in terms of a compatible coordinate system. The generalized Timoshenko Quotient and the modified Schreyer and Shih formula are the proposed upper and lower bounds. Both bounds when iterated converge to the exact buckling load. The method described here is simple and convenient and applies to all self-adjoint problems without exception.

1. INTRODUCTION

Since Weber[1] and Lord Rayleigh[2] initiated the investigation of upper bounds for eigenvalues about a century ago, numerous contributions have been made. One of the significant recent developments is probably the formulation of the minimum–maximum and maximum–minimum principles. The minimum–maximum principle by Polya[3] leads to an upper bound. The more important and difficult maximum–minimum principle by Weyl[4] leads to a lower bound. In engineering, the above two approaches are known as energy and complementary energy methods of variational principles. For the upper bound estimation, it is customary to use the Rayleigh Quotient[5]; for the lower bound, a number of formulations has been proposed[6–8]. In physics and applied mathematics, the method of intermediate problem initiated by Weinstein and improved by Aronszan, Bazley and others has proven to be surprisingly accurate[9, 10], but the method is too complex, it has not gained popularity in engineering. So the search for lower bound continues. Recently Schreyer and Shih[11] extended a method of error estimate given in [12] for the determination of a lower bound. Later, Popelar extended Schreyer and Shih's lower bound formula to elastic bodies in [13, 14].† More recently, Masur and Popelar expounded the complementary energy approach in [15] with emphasis on structural stability. The lower bound as given by Schreyer and Shih depends on both Rayleigh and Timoshenko Quotients. These quotients were defined in a way different from their usual formulation and involve boundary terms explicitly. Boundary terms are also involved in the Timoshenko Quotients formulated in [14, 15]. Accepting the basic approach of Schreyer and Shih, Ku extended the range of applicability and improved the accuracy of the lower bound formula in [16]. He further simplified the formulation of Timoshenko Quotient in [17] by eliminating the explicit boundary terms. Therefore, both Timoshenko and Rayleigh Quotients return to their familiar forms. However, the discussion in [16] was confined to uniform columns, and that the improved lower bound formula in [17] was only formally established. The purpose of the present paper is to generalize, supplement and improve the works of [16, 17]. For a systematical presentation extending to cover all the self-adjoint problems, new concepts of admissible and compatible coordinate systems may be introduced. This is discussed in the next section.

2. COMPATIBLE COORDINATE SYSTEM

A coordinate system is called admissible, if its x-axis is parallel to the undeformed axis of the column. It is obviously convenient to express both displacement and moment of the column in terms of an admissible coordinate system. Since it is unlikely that a nonparallel coordinate system will be chosen, the admissible coordinate system is not a restriction. The new term is introduced simply for convenience of discussion in the sequel. An admissible coordinate system is called compatible, if the origin of the coordinate system is selected such that the boundary

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values of the admissible displacement function satisfy the following condition

$$w(x_1)w'(x_1) = w(x_2)w'(x_2) = 0 \tag{1}$$

where x_1, x_2 are the x coordinates of the ends of the column. The conditions posed in eqn (1) is certainly not too restrictive so as to exclude the possibility of such a coordinate system. One way to establish such a coordinate system is to conceive that the support of a column is either movable laterally or the opposite. In the case of a laterally immovable end, $w = 0$ at that end, eqn (1) is satisfied. In the case of a laterally movable end, either $w' = 0$ or $w' \neq 0$ at that end; when $w' = 0$, eqn (1) is again satisfied. This leaves the movable and rotable end (or free end) as the singular case where eqn (1) may be violated if noncompatible coordinate system is used. The establishing of a compatible coordinate system for this case is illustrated by a nonuniform column in a later section. The purpose of employing a compatible coordinate system is to eliminate explicit bounding terms in the formulations.

3. TIMOSHENKO LOAD AS AN UPPER BOUND

The use of Rayleigh Quotient as an upper bound is well known. Modified Rayleigh Quotients have also been proposed[5]. On physical reasoning, Timoshenko[18] demonstrated a method to calculate the upper bound using the admissible function and its first derivative instead of the first and second derivatives as required by the Rayleigh Quotient. Although Timoshenko's method yeilds a better upper bound than the Rayleigh Quotient, it has a limited applicability as clearly stated, for example, by Simitese[19]. Generalized Timoshenko Quotients cited in the introduction are applicable without such limitation but all, without exception, involve boundary terms. To formulate a Timoshenko Quotient free from any limitations and explicit boundary terms, consider the potential energy of neutral equilibrium

$$U = \frac{1}{2}\langle kw'', kw''\rangle - \frac{P}{2}\langle w', w'\rangle \tag{2}$$

where $k^2 = EI$ is the bending rigidity of the column, w', w'' the first and second derivatives of the admissible displacement function with respect to x, and the inner product is defined as

$$\langle A, A\rangle \equiv \int_0^l A^2\,dx. \tag{3}$$

Upon setting $U = 0$ in eqn (2), the Rayleigh Quotient is obtained:

$$P_R = \frac{\langle kw'', kw''\rangle}{\langle w', w'\rangle}. \tag{4}$$

Let M be an admissible moment function compatible with the admissible displacement function w, then the complementary potential energy of neutral equilibrium is

$$U_c = \frac{1}{2}\langle fM, fM\rangle - \frac{P}{2}\langle w', w'\rangle \tag{5}$$

where $f^2 = (1/k^2)$ is the reciprocal of the bending rigidity. An admissible moment function M is said to be compatible with the admissible displacement function w, if both M and w satisfy the neutral equilibrium condition:

$$M'' - Pw'' = 0. \tag{6}$$

This equation is readily integrated to yield

$$M = Pm = P(w + ax + b). \tag{7}$$

The constants of integration a and b are determinable by boundary conditions discussed in a later section. Upon substituting eqn (7) into eqn (5) and setting $U_c = 0$, the Timoshenko Quotient is obtained

$$P_T = \frac{\langle w', w' \rangle}{\langle fm, fm \rangle}. \tag{8}$$

It will be shown that the Timoshenko Quotient given by eqn (8) is more accurate than the Rayleigh Quotient and is therefore a better upper bound.

4. A LOWER BOUND FORMULA

Although jointly the admissible moment and admissible displacement functions satisfy the equilibrium condition, the stresses or strains calculated on the basis of either admissible functions do not always agree, unless the admissible functions happen to be the eigenfunctions. The discrepancy can, therefore, serve as an indication of error. Define the residual moment by

$$M_d = k^2 w'' + Pm \tag{9}$$

and the residual complementary potential energy by

$$2\epsilon U_0 = \langle fM_d, fM_d \rangle \tag{10}$$

where $2U_0 = \langle fm, fm \rangle$ and ϵ a coefficient indicating the magnitude of the discrepancy. Upon substituting eqn (9) into eqn (10) and recognizing that

$$\langle M, w'' \rangle = -\langle w', w' \rangle \tag{11}$$

the following relation is obtained:

$$2\epsilon U_0 = \langle kw'', kw'' \rangle - 2P\langle w', w' \rangle + P^2 \langle fm, fm \rangle. \tag{12}$$

Using eqns (4) and (8), eqn (12) can be recast as

$$\epsilon = P_R P_T - 2P_T P + P^2. \tag{13}$$

The coefficient ϵ is a quadratic function of P for a specific admissible displacement function w. If w is close to the first buckling mode, for example, then P is at least meaningful in the interval $0 < P < P_2$. For lower bound estimates as will be shown later, P is at least meaningful in the interval $P_1 < P < P_2$. On the other hand, the admissible displacement and moment functions can be expanded in terms of the complete eigenfunction sets $\{W_i\}$ and $\{m_i\}$, that is:

$$\begin{aligned} w(x) &= \sum_{i=1}^{\infty} A_i W_i(x) \\ m(x) &= \sum_{i=1}^{\infty} B_i m_i(x). \end{aligned} \tag{14}$$

It can be shown that for compatible moment and displacement functions, $A_i = B_i$ $(i = 1, 2, \ldots)$. Since the eigenvalue problem under consideration is self-adjoint, the following orthnormal conditions follow:

$$\langle w'_i, w'_j \rangle = \delta_{ij} \tag{15}$$

$$\langle kw''_i, kw''_j \rangle = P_i \delta_{ij} \tag{16}$$

$$\langle fm_i, fm_j \rangle = \frac{\delta_{ij}}{P_i}. \tag{17}$$

Using these results, the following are readily established

$$\langle kw'', kw'' \rangle = \sum_{i=1}^{\infty} A_i^2 P_i \tag{18}$$

$$\langle w', w' \rangle = \sum_{i=1}^{\infty} A_i^2 \tag{19}$$

$$\langle fm, fm \rangle = \sum_{i=1}^{\infty} \frac{A_i^2}{P_i}. \tag{20}$$

Upon substituting eqns (18)–(20) into eqn (12), it yields

$$2\epsilon U_0 = \sum_{i=1}^{\infty} \frac{A_i^2 (P - P_i)^2}{P_i}. \tag{21}$$

Regarding the right-hand side of eqn (21) as a first moment of (A_i^2/P_i) with respect to $(P - P_i)^2$, it follows

$$\sum_{i=1}^{\infty} \frac{A_i^2}{P_i}(P - P_i)^2 = (P - \bar{P})^2 \sum_{i=1}^{\infty} \frac{A_i^2}{P_i}. \tag{22}$$

Since the eigenvalues are real and positive, it is possible to establish the inequality

$$(P - \bar{P})^2 \geqslant (P - P_1)^2 \tag{23}$$

provided P lies in the following interval:

$$P_1 \leqslant P \leqslant P_1^* \equiv \frac{P_R P_T - P_1^2}{2(P_T - P_1)} \tag{23a}$$

both limits of which are unknown. Using eqns (22) and (23), eqn (21) can be recast as

$$2\epsilon U_0 \geqslant (P - P_1)^2 \sum_{i=1}^{\infty} \frac{A_i^2}{P_i}. \tag{24}$$

And the inequality follows:

$$\epsilon \geqslant (P - P_1)^2 \tag{25}$$

From this inequality, a lower bound may be defined

$$P_L = P - \sqrt{\epsilon} \tag{26}$$

provided P is within the interval defined by eqn (23a). The precise boundaries of P cannot be determined *a priori*, and must be regarded as an open question. Combining eqns (26) with (13), a lower bound is obtained. Now this lower bound increases monotonically with P. With the upper boundary of the interval of P unknown, a conservative estimate of P_L is to set $P = P_R$ when it is within the aforementioned interval. The lower bound thus obtained is

$$P_L = P_R - \sqrt{[P_R(P_R - P_T)]} \tag{27}$$

which is an improvement over those reported in [11, 13]. When P is not within the said interval, say $P > P_1^*$, then eqn (26) will yield an upper bound instead of a lower bound. However, P_L is not crucially dependent on P as can be seen from the fact that $P_L \to P_T$ as $P \to \infty$; for any $P > P_1^*$, eqn (26) would still yield a better upper bound than P_T. Fortunately, for commonly encountered problems, this difficulty does not present itself.

5. PROOF OF THE TIMOSHENKO QUOTIENT AS AN UPPER BOUND

The fact that the Timoshenko Quotient is an upper bound follows from the definition of P_T and eqns (19) and (20), that is:

$$P_T = P_1 \frac{1+\sum_{i=2}^{\infty}\left(\frac{A_i}{A_1}\right)^2}{1+\sum_{i=2}^{\infty}\left(\frac{P_1}{P_i}\right)\left(\frac{A_i}{A_1}\right)^2} \geq P_1. \tag{28†}$$

The Timoshenko Quotient being a better upper bound than the Rayleigh Quotient is next proved. Consider the Cauchy–Schwarz inequality

$$\langle kw'', kw''\rangle\langle fm, fm\rangle \geq \langle kw'', fm\rangle^2. \tag{29}$$

Upon combining eqns (11) and (29) and using the definitions of P_R and P_T, it follows that

$$P_R \geq P_T \geq P_1. \tag{30}$$

Therefore, the Timoshenko Quotient is indeed a better upper bound. Although other versions of the Timoshenko Quotients must be related to the simpler one expressed by eqn (8), the relationships are by no means obvious and are not of concern here. Therefore, the proofs presented above are necessary.

6. THE CONSTRUCTION AND ITERATION OF ADMISSIBLE FUNCTIONS

From a practical point of view, once the admissible functions are known, upper and lower bounds are readily computed. The remaining question is the accuracy of the results. Therefore, it is in order to elaborate on the construction of admissible functions and to discuss the accuracy and means to improve it when required.

Any continuous function of class C^4 and satisfying geometrical boundary conditions can be considered as an admissible displacement function provided it is expressed in terms of a compatible, admissible coordinate system. A compatible admissible moment function is then constructed according to eqn (6). When the column is statically determinate, both constants a and b are readily determined from equilibrium considerations. When the column is statically indeterminate, a and b are determined by the following compatibility boundary conditions.

$$w'(l) - w'(0) = \langle f\sqrt{m}, f\sqrt{m}\rangle \tag{31}$$

$$w(l) - w(0) = w'(0) + \langle f(l-x), fm\rangle \tag{32a}$$

or

$$w(l) - w(0) = w'(l) - \langle fx, fm\rangle. \tag{32b}$$

With the admissible functions thus determined, unless they happen to be the eigenfunctions, the Rayleigh and Timoshenko Quotients involve certain amounts of error. It can be shown that the respective errors in Rayleigh and Timoshenko Quotients are Δ_R and Δ_T given below:

$$\Delta_R = A_1^2 \sum_{i=1}^{\infty} \epsilon_i^2 P_1 \left(\frac{P_i}{P_1} - 1\right) \tag{33}$$

and

$$\Delta_T = \frac{A_1^2 \sum_{i=2}^{\infty} \epsilon_i^2 P_1 \left(1 - \frac{P_1}{P_i}\right)}{1 - A_1^2 \sum_{i=2}^{\infty} \epsilon_i^2 P_1 \left(1 - \frac{P_1}{P_i}\right)} \cong A_1^2 \sum_{i=2}^{\infty} \epsilon_i^2 P_1 \left(1 - \frac{P_1}{P_i}\right) \tag{34}$$

†This is different from eqn (13) of Ref. [14].

where $\epsilon_i = A_i/A_1$ is the ratio of the expansion coefficient given in eqn (14). Although these equations do not practically provide quantitative information, meaningful qualitative information can be derived. It is seen that errors for both the Rayleigh and Timoshenko Quotients depend foremost on the closeness of the admissible displacement function to the first buckling mode. The closeness is characterized here by the ϵ_i^2's. The smaller these quantities, the closer the admissible function is to the first buckling mode. To the second order of smallness of ϵ_i, the Timoshenko Quotient is more accurate than the Rayleigh Quotient due to the scaling factors $[1-(P_1/P_i)]$ and $[(P_i/P_1)-1]$. While the scaling factor for the Timoshenko Quotient being $[1-(P_1/P_i)]<1$ and increases to the maximum value of One as $i\to\infty$, the scaling factor for the Rayleigh Quotient increases without bound as $i\to\infty$. At low i values, the magnitude of these factors depend on the difference of the corresponding eigenvalue from the first eigenvalue. Therefore, for a good selection of admissible displacement function, it is anticipated that both would yield good results. For a poor selection, the Timoshenko Quotient would yield far better results than the Rayleigh Quotient.

After the first set of admissible functions is constructed, these functions may be iterated for better accuracy. The iteration proceeds as follows. By two successive integrations of the moment curvature relationship

$$M^{(1)} = -[k^2 w^{(2)}]'' \tag{35}$$

the iterated displacement function $w^{(2)}$ is obtained. The constants of integration are made to satisfy the required boundary conditions. The iterated moment function is generated again by eqn (6) with the iterated displacement function. The process may be repeated as often as desired. It can be shown that the displacement function corresponding to the nth iteration is

$$w^{(n)} = \sum_{i=1}^{\infty} \left(\frac{P}{P_i}\right)^{n-1} A_i w_i \tag{36}$$

where $\sum_{i=1}^{\infty} A_i w_i$ is the expansion of $w^{(1)}$. As n increases indefinitely, it is clearly seen that the iterated displacement function approaches the first buckling mode. Since admissible moment function is generated by eqn (6), it must also approach the buckling moment function. As a first example, consider the buckling of a uniform column with both ends fixed. The compatible coordinate system in this case is not unique, any admissible coordinate system would be acceptable. The iterated moment and displacement functions along with upper and lower bounds are shown in Table 1.

As a second example, consider the buckling of a nonuniform cantilever column. The compatible coordinate system for this problem requires the origin of the coordinate system be located at the deflected free end. In terms of this coordinate system, $w(0) = w'(0) = 0$, eqn (1) is satisfied. Assume the bending rigidity of the column varies according to $k_\xi^2 = k_\alpha^2(\xi/\alpha)^2$, $\alpha \le \xi \equiv x/l \le 1+\alpha$ where $\alpha = 1/\sqrt{(2)}-1$. The exact buckling load is $P_1 = 4.046(k_\alpha/l)^2$[18]. Using the following admissible moment and displacement function

$$m = \rho w = \rho[\xi^2 - 2(1+\alpha)\xi + \alpha(2+\alpha)] \tag{37}$$

the Rayleigh Quotient, the Timoshenko Quotient and the lower bound are calculated as:

$$P_R = 4.4141\left(\frac{k_\alpha}{l}\right)^2 \tag{38}$$

Table 1. Upper and lower bounds

Iteration No.	0	1
Displacement function	$a(30\xi^4 - 60\xi^3 + 30\xi^2)$	$b(4\xi^6 - 12\xi^5 + 10\xi^4 - 2\xi^2)$
Moment function	$aP(30\xi^4 - 60\xi^3 + 30\xi^2 - 1)$	$bP(4\xi^6 - 12\xi^5 + 10\xi^4 - 2\xi^2 + 2/21)$
Upper bound coefficient	40	39.517
Lower bound coefficient	32.835	37.691
Exact	$4\pi^2 = 39.478$	

$$P_T = 4.065\left(\frac{k_\alpha}{l}\right)^2 \tag{39}$$

$$P_L = 3.173\left(\frac{k_\alpha}{l}\right)^2. \tag{40}$$

7. CONCLUSIONS

The Rayleigh Quotient as an upper bound possesses a number of appealing points. It is easy to apply and applies without exception to all self-adjoint problems. Although several versions of the Timoshenko Quotients are now available[11, 13–15], as far as column stability is concerned, the version presented in this paper clearly demonstrates that it is comparable to the Rayleigh Quotient in terms of the aforementioned appealing points. Since the Timoshenko Quotient is a better upper bound, the little additional effort required to obtain it is amply rewarded by the lower bound results derived from it. The lower bound formula presented in the present paper is a considerable improvement of those reported in Refs. [11, 13]. However, the optimal lower bound remains an open question and there is a possibility that eqn (26) may fail. In that event it provides a better upper bound than the Timoshenko Quotient. Finally, both the upper and lower bounds converge to the exact buckling load when the moment and displacement functions are iterated.

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